Hypergeometric type operators and their supersymmetric partners

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The generalization of the factorization method performed by Mielnik [J. Math. Phys. **25**, 3387 (1984)] opened new ways to generate exactly solvable potentials in quantum mechanics. We present an application of Mielnik's method to hypergeometric type operators. It is based on some solvable Riccati equations and leads to a unitary description of the quantum systems exactly solvable in terms of orthogonal polynomials or associated special functions.

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I. INTRODUCTION

Most of the exactly solvable Schrödinger equations are directly related to some hypergeometric type equations and most of the factorizations used in quantum mechanics[1–4] can be obtained from factorizations concerning the hypergeometric type operators. The use of the factorization method at the deeper level of hypergeometric type operators allows us to analyse together[5–8] almost all the known exactly solvable quantum systems. The unitary view obtained in this way allows us to generalize certain results known in particular cases and a transfer of ideas and methods among quantum systems. In the first part of the paper we review in a form suitable for our purpose several results concerning the orthogonal polynomials, associated special functions and hypergeometric type operators. Particularly, we present some factorizations of the hypergeometric type operators leading to particular solutions for the Riccati equations we use in the second part of the paper. Our main purpose is to present an application of Mielnik's method to hypergeometric type operators. The implementation of this method directly at the level of hypergeometric type operators allows us to enlarge our unitary view on the exactly solvable quantum systems.

Many problems in quantum mechanics and mathematical physics lead to equations of the type

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \tag{1}$$

where $\sigma(s)$ and $\tau(s)$ are polynomials of at most second and first degree, respectively, and λ is a constant. These equations are usually called *equations of hypergeometric type*[9], and each of them can be reduced to the self-adjoint form

$$[\sigma(s)\varrho(s)y'(s)]' + \lambda\varrho(s)y(s) = 0 \tag{2}$$

where

$$\varrho(s) = \frac{1}{\sigma(s)}\,\mathrm{e}^{\int^s \frac{\tau(t)}{\sigma(t)}\,dt}. \tag{3}$$

The equation (1) is usually considered on an interval (a,b), chosen such that

$$\begin{array}{c}\sigma(s) > 0 \qquad \text{for all} \quad s \in (a,b)\\ \varrho(s) > 0 \qquad \text{for all} \quad s \in (a,b)\\ \lim_{s\to a}\sigma(s)\varrho(s) = \lim_{s\to b}\sigma(s)\varrho(s) = 0.\end{array} \tag{4}$$

TABLE I. The main cases

$\sigma(s)$	$\tau(s)$	$\varrho(s)$	(a,b)	α,β
1	$\alpha s+\beta$	$\mathrm{e}^{\alpha s^2/2+\beta s}$	$(-\infty,\infty)$	$\alpha<0$
s	$\alpha s+\beta$	$s^{\beta-1}\mathrm{e}^{\alpha s}$	$(0,\infty)$	$\alpha<0,\ \beta>0$
$1-s^2$	$\alpha s+\beta$	$(1+s)^{-(\alpha-\beta)/2-1}(1-s)^{-(\alpha+\beta)/2-1}$	$(-1,1)$	$\alpha<\beta<-\alpha$
s^2-1	$\alpha s+\beta$	$(s+1)^{(\alpha-\beta)/2-1}(s-1)^{(\alpha+\beta)/2-1}$	$(1,\infty)$	$-\beta<\alpha<0$
s^2	$\alpha s+\beta$	$s^{\alpha-2}\mathrm{e}^{-\beta/s}$	$(0,\infty)$	$\alpha<0,\ \beta>0$
s^2+1	$\alpha s+\beta$	$(1+s^2)^{\alpha/2-1}\mathrm{e}^{\beta\arctan s}$	$(-\infty,\infty)$	$\alpha<0$

Since the form of the equation (1) is invariant under a change of variable $s\mapsto cs+d$, it is sufficient to analyze the cases presented in Table 1. Some restrictions are imposed on α and β in order that the interval (a,b) exist.

II. HYPERGEOMETRIC TYPE OPERATORS

The equation (1) admits for $\lambda=\lambda_\ell$ with $\ell\in\mathbb{N}$ and

$$\lambda_\ell=-\frac{\sigma''(s)}{2}\,\ell(\ell-1)-\tau'(s)\,\ell \tag{5}$$

a polynomial solution $\Phi_\ell=\Phi_\ell^{(\alpha,\beta)}$ of at most ℓ degree

$$\sigma(s)\Phi_\ell''+\tau(s)\Phi_\ell'+\lambda_\ell\Phi_\ell=0. \tag{6}$$

The function $\Phi_\ell(s)\sqrt{\varrho(s)}$ is square integrable on (a,b) and $0=\lambda_0<\lambda_1<\lambda_2<\ \dots\ <\lambda_\ell$ for any $\ell<\Lambda$, where $\Lambda=\infty$ if $\sigma\in\{1,\ s,\ 1-s^2\}$ and $\Lambda=(1-\alpha)/2$ if $\sigma\in\{s^2-1,\ s^2,\ s^2+1\}$. The system of polynomials $\{\Phi_\ell\mid\ell<\Lambda\}$ is orthogonal with weight function $\varrho(s)$ in (a,b), and Φ_ℓ is a polynomial of degree ℓ for any $\ell<\Lambda$. The polynomials Φ_ℓ can be described by using

the classical orthogonal polynomials. Up to a multiplicative constant[6,7]

$$\Phi_\ell^{(\alpha,\beta)}(s)=\begin{cases} H_\ell\left(\sqrt{\frac{-\alpha}{2}}\,s-\frac{\beta}{\sqrt{-2\alpha}}\right) & \text{if}\quad \sigma(s)=1\\ L_\ell^{\beta-1}(-\alpha s) & \text{if}\quad \sigma(s)=s\\ P_\ell^{(-(\alpha+\beta)/2-1,\ (-\alpha+\beta)/2-1)}(s) & \text{if}\quad \sigma(s)=1-s^2\\ P_\ell^{((\alpha-\beta)/2-1,\ (\alpha+\beta)/2-1)}(-s) & \text{if}\quad \sigma(s)=s^2-1\\ \left(\frac{s}{\beta}\right)^\ell L_\ell^{1-\alpha-2l}\left(\frac{\beta}{s}\right) & \text{if}\quad \sigma(s)=s^2\\ \mathrm{i}^\ell P_\ell^{((\alpha+\mathrm{i}\beta)/2-1,\ (\alpha-\mathrm{i}\beta)/2-1)}(\mathrm{i}s) & \text{if}\quad \sigma(s)=s^2+1 \end{cases} \tag{7}$$

where H_ℓ, L_ℓ^p and $P_\ell^{(p,q)}$ are the Hermite, Laguerre and Jacobi polynomials, respectively. One can remark that the relation (7) does not have a very simple form. In certain cases we have to consider the classical polynomials outside the interval where they are orthogonal or for complex values of parameters.

Let $\ell\in\mathbb{N}$, $\ell<\Lambda$, and let $m\in\{0,1,...,\ell\}$. If we differentiate (6) m times then we get

$$\sigma(s)\frac{\mathrm{d}^{m+2}}{\mathrm{d}s^{m+2}}\Phi_\ell+[\tau(s)+m\sigma'(s)]\frac{\mathrm{d}^{m+1}}{\mathrm{d}s^{m+1}}\Phi_\ell+(\lambda_\ell-\lambda_m)\frac{\mathrm{d}^m}{\mathrm{d}s^m}\Phi_\ell=0. \tag{8}$$

The equation obtained by multiplying this relation by $\sqrt{\sigma^m(s)}$ can be written as

$$\mathcal{H}_m\Phi_{\ell,m}=\lambda_l\Phi_{\ell,m} \tag{9}$$

where $\mathcal{H}_m$ is the hypergeometric type operator

$$\begin{aligned}\mathcal{H}_m=-\sigma(s)\tfrac{d^2}{ds^2}-\tau(s)\tfrac{d}{ds}+\tfrac{m(m-2)}{4}\tfrac{(\sigma'(s))^2}{\sigma(s)}+\tfrac{m\tau(s)}{2}\tfrac{\sigma'(s)}{\sigma(s)}\\ -\tfrac{1}{2}m(m-2)\sigma''(s)-m\tau'(s)\end{aligned} \tag{10}$$

and the functions

$$\Phi_{\ell,m}(s)=\kappa^m(s)\frac{\mathrm{d}^m}{\mathrm{d}s^m}\Phi_\ell(s) \tag{11}$$

defined by using

$$\kappa(s)=\sqrt{\sigma(s)} \tag{12}$$

are called the *associated special functions.*

III. SOME PARTICULAR FACTORIZATIONS

By differentiating (6) $m-1$ times we obtain

$$\sigma(s)\tfrac{\mathrm{d}^{m+1}}{\mathrm{d}s^{m+1}}\Phi_\ell(s)+(m-1)\sigma'(s)\tfrac{\mathrm{d}^{m}}{\mathrm{d}s^{m}}\Phi_\ell(s)+\tfrac{(m-1)(m-2)}{2}\sigma''(s)\tfrac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}}\Phi_\ell(s)$$
$$+\tau(s)\tfrac{\mathrm{d}^{m}}{\mathrm{d}s^{m}}\Phi_\ell(s)+(m-1)\tau'(s)\tfrac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}}\Phi_\ell(s)+\lambda_\ell\tfrac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}}\Phi_\ell(s)=0.$$

If we multiply this relation by $\kappa^{m-1}(s)$, then we get the three term recurrence relation

$$\Phi_{\ell,m+1}(s)+\left(\tfrac{\tau(s)}{\kappa(s)}+2(m-1)\kappa'(s)\right)\Phi_{\ell,m}(s)+(\lambda_\ell-\lambda_{m-1})\Phi_{\ell,m-1}(s)=0 \tag{13}$$

for $m\in\{1,2,...,\ell-1\}$, and

$$\left(\frac{\tau(s)}{\kappa(s)}+2(\ell-1)\kappa'(s)\right)\Phi_{\ell,\ell}(s)+(\lambda_\ell-\lambda_{\ell-1})\Phi_{\ell,\ell-1}(s)=0 \tag{14}$$

for $m=\ell$. For each $m\in\{0,1,...,\ell-1\}$, by differentiating (11), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi_{\ell,m}(s)=m\kappa^{m-1}(s)\,\kappa'(s)\frac{\mathrm{d}^m}{\mathrm{d}s^m}\Phi_\ell(s)+\kappa^m(s)\frac{\mathrm{d}^{m+1}}{\mathrm{d}s^{m+1}}\Phi_\ell(s)$$

that is, the relation

$$\kappa(s)\frac{\mathrm{d}}{\mathrm{d}s}\Phi_{\ell,m}(s)=m\kappa'(s)\Phi_{\ell,m}(s)+\Phi_{\ell,m+1}(s)$$

which can be written as

$$\left(\kappa(s)\frac{d}{ds}-m\kappa'(s)\right)\Phi_{\ell,m}(s)=\Phi_{\ell,m+1}(s). \tag{15}$$

If $m\in\{1,2,...,\ell-1\}$, then by substituting (15) into (13), we get

$$\left(\kappa(s)\frac{d}{ds}+\frac{\tau(s)}{\kappa(s)}+(m-2)\kappa'(s)\right)\Phi_{\ell,m}(s)+(\lambda_\ell-\lambda_{m-1})\Phi_{\ell,m-1}(s)=0$$

that is,

$$\left(-\kappa(s)\frac{d}{ds}-\frac{\tau(s)}{\kappa(s)}-(m-1)\kappa'(s)\right)\Phi_{\ell,m+1}(s)=(\lambda_\ell-\lambda_m)\Phi_{\ell,m}(s). \tag{16}$$

for all $m\in\{0,1,...,\ell-2\}$. From (14) it follows that this relation is also satisfied for $m=\ell-1$. The relations (15) and (16) suggest we consider for $m+1<\Lambda$ the operators[5–8]

$$\begin{aligned} a_m &= \kappa(s)\left(\tfrac{d}{ds}-m\tfrac{\kappa'(s)}{\kappa(s)}\right)\\ a_m^+ &= \kappa(s)\left(-\tfrac{d}{ds}-\tfrac{\tau(s)}{\sigma(s)}-(m-1)\tfrac{\kappa'(s)}{\kappa(s)}\right)\end{aligned} \tag{17}$$

satisfying the relations

$$a_m\Phi_{\ell,m} = \begin{cases} 0 & \text{for } \ \ell = m \\ \Phi_{\ell,m+1} & \text{for } \ m < \ell < \Lambda \end{cases} \tag{18}$$
$$a_m^+\Phi_{\ell,m+1} = (\lambda_\ell - \lambda_m)\Phi_{\ell,m} \ \text{ for } \ 0 \leq m < \ell < \Lambda.$$

and

$$\Phi_{\ell,m}(s) = \begin{cases} \kappa^\ell(s) & \text{for} \quad m = \ell \\ \frac{a_m^+}{\lambda_\ell-\lambda_m}\frac{a_{m+1}^+}{\lambda_\ell-\lambda_{m+1}}\cdots\frac{a_{\ell-1}^+}{\lambda_\ell-\lambda_{\ell-1}}\kappa^\ell(s) & \text{for} \quad 0 < m < \ell < \Lambda. \end{cases} \tag{19}$$

For each $m < \Lambda$, the functions $\Phi_{\ell,m}$ with $m \leq \ell < \Lambda$ are orthogonal with weight function $\varrho(s)$ in (a,b). If $0 \leq m < \ell < \Lambda$ then $||\Phi_{\ell,m+1}|| = \sqrt{\lambda_\ell - \lambda_m}\, ||\Phi_{\ell,m}||$. The operators $\mathcal{H}_m$ can be expressed in terms of the functions $v_m : (a,b) \longrightarrow \mathbb{R}$,

$$v_m(s) = \frac{m(m-2)}{4}\frac{(\sigma'(s))^2}{\sigma(s)} + \frac{m\tau(s)}{2}\frac{\sigma'(s)}{\sigma(s)} - \frac{1}{2}m(m-2)\sigma''(s) - m\tau'(s) \tag{20}$$

as

$$\mathcal{H}_m = -\sigma(s)\frac{d^2}{ds^2} - \tau(s)\frac{d}{ds} + v_m(s). \tag{21}$$

They are shape invariant

$$\mathcal{H}_m - \lambda_m = a_m^+ a_m \qquad \mathcal{H}_{m+1} - \lambda_m = a_m a_m^+ \tag{22}$$

and satisfy the intertwining relations

$$\mathcal{H}_m a_m^+ = a_m^+ \mathcal{H}_{m+1} \qquad a_m \mathcal{H}_m = \mathcal{H}_{m+1} a_m. \tag{23}$$

If α and β are such that $\varrho(s) = \sigma^k(s)$ (see table 2) then the operators

$$\tilde{\mathcal{H}}_m = \mathcal{H}_m - \delta\,\kappa'(s) = -\sigma(s)\frac{d^2}{ds^2} - \tau(s)\frac{d}{ds} + \tilde{v}_m(s) \tag{24}$$

admit for $m < \Lambda - 1$ with $2m + 2k + 1 \neq 0$ and any $\delta \in \mathbb{R}$ the factorizations[7]

$$\tilde{\mathcal{H}}_m - \tilde{\lambda}_m = \tilde{a}_m^+ \tilde{a}_m, \qquad \tilde{\mathcal{H}}_{m+1} - \tilde{\lambda}_m = \tilde{a}_m \tilde{a}_m^+ \tag{25}$$

where

$$\tilde{a}_m = \kappa(s)\left(\frac{d}{ds} - m\frac{\kappa'(s)}{\kappa(s)}\right) + \frac{\delta}{2m+2k+1}$$
$$\tilde{a}_m^+ = \kappa(s)\left(-\frac{d}{ds} - \frac{\tau(s)}{\sigma(s)} - (m-1)\frac{\kappa'(s)}{\kappa(s)}\right) + \frac{\delta}{2m+2k+1} \tag{26}$$

and

$$\tilde{\lambda}_m = \lambda_m - \frac{\delta^2}{(2m+2k+1)^2}. \tag{27}$$

TABLE II. The cases when $\varrho(s)=\sigma^k(s)$.

$\sigma(s)$	$\tau(s)$	$\varrho(s)$	(a,b)	k
s	β	$s^{\beta-1}$	$(0,\infty)$	$\beta-1$
$1-s^2$	αs	$(1-s^2)^{-\alpha/2-1}$	$(-1,1)$	$-\frac{\alpha}{2}-1$
s^2-1	αs	$(s^2-1)^{\alpha/2-1}$	$(1,\infty)$	$\frac{\alpha}{2}-1$
s^2	αs	$s^{\alpha-2}$	$(0,\infty)$	$\frac{\alpha}{2}-1$
s^2+1	αs	$(s^2+1)^{\alpha/2-1}$	$(-\infty,\infty)$	$\frac{\alpha}{2}-1$

IV. SUPERSYMMETRIC PARTNERS

By following Mielnik's idea[4], we look for a more general factorization

$$\mathcal{H}_{m+1}-\lambda_m=b_m b_m^+ \tag{28}$$

with b_m and b_m^+ of the form

$$b_m=\kappa(s)\left(\frac{d}{ds}+\varphi(s)\right), \qquad b_m^+=\kappa(s)\left(-\frac{d}{ds}+\psi(s)\right). \tag{29}$$

In order to get the factorization (28), the function φ must satisfy the relation

$$\varphi(s)=\psi(s)+\frac{\tau(s)}{\sigma(s)}-\frac{\kappa'(s)}{\kappa(s)} \tag{30}$$

and ψ be a solution of the Riccati equation

$$\psi'=-\psi^2-\frac{\tau(s)}{\sigma(s)}\,\psi+\frac{v_{m+1}(s)-\lambda_m}{\sigma(s)}. \tag{31}$$

In view of (17) and (22) this equation has the particular solution

$$\psi(s)=-\frac{\tau(s)}{\sigma(s)}-(m-1)\frac{\kappa'(s)}{\kappa(s)}=-\frac{\tau(s)}{\sigma(s)}-\frac{m-1}{2}\frac{\sigma'(s)}{\sigma(s)}. \tag{32}$$

The general solution is (see (3))

$$\psi_\gamma(s)=-\frac{\tau(s)}{\sigma(s)}-\frac{m-1}{2}\frac{\sigma'(s)}{\sigma(s)}+\frac{\sigma^m(s)\,\varrho(s)}{\gamma+\int^s\sigma^m(t)\,\varrho(t)dt} \tag{33}$$

where γ is a constant such that ψ has no singularity. The operators

$$\begin{aligned}\mathcal{H}_{m,\gamma}&=b_m^+\,b_m+\lambda_m\\&=\kappa(s)\left(-\tfrac{d}{ds}+\psi_\gamma(s)\right)\ \kappa(s)\left(\tfrac{d}{ds}+\varphi_\gamma(s)\right)+\lambda_m\end{aligned} \tag{34}$$

where

$$\varphi_\gamma(s) = -\frac{m}{2}\frac{\sigma'(s)}{\sigma(s)} + \frac{\sigma^m(s)\,\varrho(s)}{\gamma + \int^s \sigma^m(t)\,\varrho(t)dt} \tag{35}$$

have the form

$$\mathcal{H}_{m,\gamma} = -\sigma(s)\frac{d^2}{ds^2} - \tau(s)\frac{d}{ds} + v_{m,\gamma}(s) \tag{36}$$

and can be regarded as 'supersymmetric' partners of $\mathcal{H}_{m+1}$. The eigenfunctions of the operators $\mathcal{H}_{m,\gamma}$ are directly related to the special functions $\Phi_{\ell,m+1}$, namely, we have

$$\mathcal{H}_{m,\gamma}\,(b_m^+\Phi_{\ell,m+1}) = (b_m^+\,b_m + \lambda_m)b_m^+\Phi_{\ell,m+1} = b_m^+\,\mathcal{H}_{m+1}\,\Phi_{\ell,m+1} = \lambda_\ell\,(b_m^+\Phi_{\ell,m+1}). \tag{37}$$

Example. In the case $\sigma(s)=1$, $\tau(s)=\alpha s+\beta$ (see table 1) the operator

$$\mathcal{H}_{m+1} = -\frac{d^2}{ds^2} - (\alpha s+\beta)\frac{d}{ds} - \alpha m \tag{38}$$

admits the supersymmetric partners

$$\mathcal{H}_{m,\gamma} = \left(-\frac{d}{ds}+\psi_\gamma(s)\right)\,\left(\frac{d}{ds}+\varphi_\gamma(s)\right) - \alpha m \tag{39}$$

where

$$\psi_\gamma(s) = -(\alpha s+\beta) + \frac{\mathrm{e}^{\alpha\frac{s^2}{2}+\beta s}}{\gamma + \int_0^s \mathrm{e}^{\alpha\frac{t^2}{2}+\beta t}dt} \tag{40}$$

and

$$\varphi_\gamma(s) = \frac{\mathrm{e}^{\alpha\frac{s^2}{2}+\beta s}}{\gamma + \int_0^s \mathrm{e}^{\alpha\frac{t^2}{2}+\beta t}dt}. \tag{41}$$

Particularly, for $\alpha=-2$, $\beta=0$ and $m=0$ the operator

$$\mathcal{H}_1 = -\frac{d^2}{ds^2} + 2s\frac{d}{ds} \tag{42}$$

admits the supersymmetric partners

$$\mathcal{H}_{0,\gamma} = \left(-\frac{d}{ds}+2s+\frac{\mathrm{e}^{-s^2}}{\gamma+\int_0^s \mathrm{e}^{-t^2}dt}\right)\left(\frac{d}{ds}+\frac{\mathrm{e}^{-s^2}}{\gamma+\int_0^s \mathrm{e}^{-t^2}dt}\right). \tag{43}$$

If α and β are such that $\varrho(s)=\sigma^k(s)$ (see table 2) then the operator

$$\tilde{\mathcal{H}}_{m+1} = \mathcal{H}_{m+1} - \delta\,\kappa'(s) = \tilde{b}_m\,\tilde{b}_m^+ + \tilde{\lambda}_m \tag{44}$$

admits for $m<\Lambda-1$ with $2m+2k+1\neq 0$ and any $\delta\in\mathbb{R}$ the supersymmetric partners

$$\tilde{\mathcal{H}}_{m,\gamma} = \tilde{b}_m^+\,\tilde{b}_m + \tilde{\lambda}_m \tag{45}$$

where

$$\begin{aligned}\tilde{b}_m &= \kappa(s)\left(\tfrac{d}{ds}+\varphi_\gamma(s)\right) + \tfrac{\delta}{2m+2k+1}\\ \tilde{b}_m^+ &= \kappa(s)\left(-\tfrac{d}{ds}+\psi_\gamma(s)\right) + \tfrac{\delta}{2m+2k+1}.\end{aligned} \tag{46}$$

V. SCHRÖDINGER TYPE OPERATORS

If we apply in $\mathcal{H}_m\Phi_{\ell,m}=\lambda_\ell\Phi_{\ell,m}$ a change of variable $(a',b')\to(a,b):x\mapsto s(x)$ such that $ds/dx=\kappa(s(x))$ or $ds/dx=-\kappa(s(x))$ and define the new functions

$$\Psi_{\ell,m}(x)=\sqrt{\kappa(s(x))\,\varrho(s(x))}\,\Phi_{\ell,m}(s(x)) \tag{47}$$

then we get an equation of Schrödinger type[1,8]

$$-\frac{d^2}{dx^2}\Psi_{\ell,m}(x)+V_m(x)\Psi_{\ell,m}(x)=\lambda_\ell\Psi_{\ell,m}(x). \tag{48}$$

If $ds/dx=\pm\kappa(s(x))$ then the operators corresponding to b_m and b_m^+ are

$$\begin{aligned} B_m&=[\kappa(s)\varrho(s)]^{1/2}b_m[\kappa(s)\varrho(s)]^{-1/2}|_{s=s(x)}=\pm\tfrac{d}{dx}+W_{m,\gamma}(x)\\ B_m^+&=[\kappa(s)\varrho(s)]^{1/2}b_m^+[\kappa(s)\varrho(s)]^{-1/2}|_{s=s(x)}=\mp\tfrac{d}{dx}+W_{m,\gamma}(x)\end{aligned} \tag{49}$$

where the *superpotential* $W_{m,\gamma}(x)$ is given by the formula

$$W_{m,\gamma}(x)=-\frac{\tau(s(x))}{2\kappa(s(x))}-\left(m-\frac{1}{2}\right)\kappa'(s(x))+\kappa(s(x))\,\frac{\sigma^m(s(x))\ \varrho(s(x))}{\gamma+\int^{s(x)}\sigma^m(t)\ \varrho(t)dt}\,. \tag{50}$$

The operator corresponding to $\mathcal{H}_{m+1}$, namely,

$$-\frac{d^2}{dx^2}+V_{m+1}(x)=B_mB_m^++\lambda_m=-\frac{d^2}{dx^2}+W_{m,\gamma}^2(x)\pm W_{m,\gamma}'(x)+\lambda_m \tag{51}$$

admits the supersymmetric partners

$$-\frac{d^2}{dx^2}+V_{m,\gamma}(x)=B_m^+B_m+\lambda_m=-\frac{d^2}{dx^2}+W_{m,\gamma}^2(x)\mp W_{m,\gamma}'(x)+\lambda_m \tag{52}$$

and

$$\left(-\frac{d^2}{dx^2}+V_{m,\gamma}(x)\right)(B_m^+\Psi_{\ell,m+1})=\lambda_l(B_m^+\Psi_{\ell,m+1}). \tag{53}$$

If α and β are such that $\varrho(s)=\sigma^k(s)$ then the operators corresponding to $\tilde{b}_m$ and $\tilde{b}_m^+$ are

$$\begin{aligned} \tilde{B}_m&=[\kappa(s)\varrho(s)]^{1/2}\tilde{b}_m[\kappa(s)\varrho(s)]^{-1/2}|_{s=s(x)}=\pm\tfrac{d}{dx}+\tilde{W}_{m,\gamma}(x)\\ \tilde{B}_m^+&=[\kappa(s)\varrho(s)]^{1/2}\tilde{b}_m^+[\kappa(s)\varrho(s)]^{-1/2}|_{s=s(x)}=\mp\tfrac{d}{dx}+\tilde{W}_{m,\gamma}(x)\end{aligned} \tag{54}$$

where the *superpotential* $\tilde{W}_{m,\gamma}(x)$ is given by the formula

$$\tilde{W}_{m,\gamma}(x)=-\tfrac{\tau(s(x))}{2\kappa(s(x))}-\left(m-\tfrac{1}{2}\right)\kappa'(s(x))+\kappa(s(x))\,\tfrac{\sigma^m(s(x))\ \varrho(s(x))}{\gamma+\int^{s(x)}\sigma^m(t)\ \varrho(t)dt}+\tfrac{\delta}{2m+2k+1}\,. \tag{55}$$

The operator corresponding to $\tilde{\mathcal{H}}_{m+1}$, namely,

$$-\frac{d^2}{dx^2}+\tilde{V}_{m+1}(x)=\tilde{B}_m\tilde{B}_m^+ +\tilde{\lambda}_m=-\frac{d^2}{dx^2}+\tilde{W}_{m,\gamma}^2(x)\pm\tilde{W}'_{m,\gamma}(x)+\tilde{\lambda}_m \tag{56}$$

admits the supersymmetric partners

$$-\frac{d^2}{dx^2}+\tilde{V}_{m,\gamma}(x)=\tilde{B}_m^+\tilde{B}_m+\tilde{\lambda}_m=-\frac{d^2}{dx^2}+\tilde{W}_{m,\gamma}^2(x)\mp\tilde{W}'_{m,\gamma}(x)+\tilde{\lambda}_m. \tag{57}$$

Examples. Let $\alpha_m=-(2m+\alpha-1)/2$ and $\alpha'_m=(2m-\alpha-1)/2$.

1. *Shifted oscillator*

   In the case $\sigma(s)=1$, $\tau(s)=\alpha s+\beta$, the change of variable $s(x)=x$ leads to

$$\begin{aligned} V_{m+1}(x) &= \tfrac{(\alpha x+\beta)^2}{4}-\tfrac{\alpha}{2}+\lambda_m \\ W_{m,\gamma}(x) &= -\tfrac{\alpha x+\beta}{2}+\frac{\mathrm{e}^{\alpha\frac{x^2}{2}+\beta x}}{\gamma+\int_0^x \mathrm{e}^{\alpha\frac{t^2}{2}+\beta t}dt} \\ \lambda_m &= -\alpha m. \end{aligned}$$

   Particularly, for $\alpha=-2$, $\beta=0$ we get

$$W_{m,\gamma}(x)=x+\frac{\mathrm{e}^{-x^2}}{\gamma+\int_0^x \mathrm{e}^{-t^2}dt}.$$

2. *Three-dimensional oscillator*

   In the case $\sigma(s)=s$, $\tau(s)=\alpha s+\beta$, the change of variable $s(x)=x^2/4$ leads to

$$\begin{aligned} V_{m+1}(x) &= \tfrac{\alpha^2}{16}x^2+\left(\beta+m-\tfrac{1}{2}\right)\left(\beta+m+\tfrac{1}{2}\right)\tfrac{1}{x^2}+\tfrac{\alpha}{2}(\beta+m-1)+\lambda_m \\ W_{m,\gamma}(x) &= -\tfrac{\alpha}{4}x-\left(\beta+m-\tfrac{1}{2}\right)\tfrac{1}{x}+\tfrac{1}{2^{2m+2\beta-1}}\frac{x^{2m+2\beta-1}\,\mathrm{e}^{\alpha x^2/4}}{\gamma+\int^{x^2/4} t^{m+\beta-1}\,\mathrm{e}^{\alpha t}dt} \\ \lambda_m &= -\alpha m. \end{aligned}$$

3. *Pöschl-Teller type potential*

   In the case $\sigma(s)=1-s^2$, $\tau(s)=\alpha s+\beta$, the change of variable $s(x)=\cos x$ leads to

$$\begin{aligned} V_{m+1}(x) &= \left({\alpha'_m}^2+\alpha'_m+\tfrac{\beta^2}{4}\right)\mathrm{cosec}^2 x-(2\alpha'_m+1)\tfrac{\beta}{2}\,\mathrm{cotan}\,x\ \mathrm{cosec}\,x-{\alpha'_m}^2+\lambda_m \\ W_{m,\gamma}(x) &= \alpha'_m\mathrm{cotan}\,x-\tfrac{\beta}{2}\,\mathrm{cosec}\,x+\sin x\,\frac{(1+\cos x)^{-(\alpha-\beta)/2+m-1}(1-\cos x)^{-(\alpha+\beta)/2+m-1}}{\gamma+\int^{\cos x}(1+t)^{-(\alpha-\beta)/2+m-1}(1-t)^{-(\alpha+\beta)/2+m-1}dt} \\ \lambda_m &= m(m-\alpha-1). \end{aligned}$$

4. *Generalized Pöschl-Teller potential*

   In the case $\sigma(s)=s^2-1$, $\tau(s)=\alpha s+\beta$, the change of variable $s(x)=\cosh x$ leads to

$$\begin{aligned} V_{m+1}(x) &= \left(\alpha_m^2-\alpha_m+\tfrac{\beta^2}{4}\right)\mathrm{cosech}^2 x-(2\alpha_m-1)\tfrac{\beta}{2}\,\mathrm{cotanh}\,x\ \mathrm{cosech}\,x+\alpha_m^2+\lambda_m \\ W_{m,\gamma}(x) &= \alpha_m\mathrm{cotanh}\,x-\tfrac{\beta}{2}\,\mathrm{cosech}\,x+\sinh x\,\frac{(\cosh x+1)^{(\alpha-\beta)/2+m-1}(\cosh x-1)^{(\alpha+\beta)/2+m-1}}{\gamma+\int^{\cosh x}(t+1)^{(\alpha-\beta)/2+m-1}(t-1)^{(\alpha+\beta)/2+m-1}dt} \\ \lambda_m &= -m(m+\alpha-1). \end{aligned}$$

5. *Morse type potential*

In the case $\sigma(s) = s^2$, $\tau(s) = \alpha s + \beta$, the change of variable $s(x) = \mathrm{e}^x$ leads to

$$
\begin{aligned}
V_{m+1}(x) &= \tfrac{\beta^2}{4}\mathrm{e}^{-2x} - (2\alpha_m - 1)\tfrac{\beta}{2}\mathrm{e}^{-x} + \alpha_m^2 + \lambda_m \\
W_{m,\gamma}(x)(x) &= -\tfrac{\beta}{2}\mathrm{e}^{-x} + \alpha_m + \mathrm{e}^x \tfrac{\mathrm{e}^{(2m+\alpha-2)x}\,\mathrm{e}^{-\beta\mathrm{e}^{-x}}}{\gamma + \int^{\mathrm{e}^x} t^{2m+\alpha-2}\mathrm{e}^{-\beta/t}dt} \\
\lambda_m &= -m(m+\alpha-1).
\end{aligned}
$$

6. *Scarf hyperbolic type potential*

In the case $\sigma(s) = s^2 + 1$, $\tau(s) = \alpha s + \beta$, the change of variable $s(x) = \sinh x$ leads to

$$
\begin{aligned}
V_{m+1}(x) &= \left(-\alpha_m^2 + \alpha_m + \tfrac{\beta^2}{4}\right)\operatorname{sech}^2 x - (2\alpha_m - 1)\tfrac{\beta}{2}\tanh x \,\operatorname{sech} x + \alpha_m^2 + \lambda_m \\
W_{m,\gamma}(x) &= \alpha_m \tanh x - \tfrac{\beta}{2}\operatorname{sech} x + \cosh x \,\tfrac{(\cosh x)^{2m+\alpha-2}\mathrm{e}^{\beta\arctan(\sinh \mathrm{x})}}{\gamma + \int^{\sinh x}(t^2+1)^{\alpha/2+m-1}\mathrm{e}^{\beta\arctan t}dt} \\
\lambda_m &= -m(m+\alpha-1).
\end{aligned}
$$

7. *Coulomb type potential*

In the case $\sigma(s) = s$, $\tau(s) = \beta$, the change of variable $s(x) = x^2/4$ leads to

$$
\begin{aligned}
\tilde{V}_{m+1}(x) &= \left(\beta + m - \tfrac{1}{2}\right)\left(\beta + m + \tfrac{1}{2}\right)\tfrac{1}{x^2} - \delta\tfrac{1}{x} \\
\tilde{W}_{m,\gamma}(x) &= -\left(\beta + m - \tfrac{1}{2}\right)\tfrac{1}{x} + \tfrac{1}{2^{2m+2\beta-1}}\tfrac{x^{2m+2\beta-1}}{\gamma + \int^{x^2/4} t^{m+\beta-1}dt} + \tfrac{\delta}{2m+2\beta-1} \\
\tilde{\lambda}_m &= -\tfrac{\delta^2}{(2m+2\beta-1)^2}.
\end{aligned}
$$

8. *Trigonometric Rosen-Morse type potential*

In the case $\sigma(s) = 1 - s^2$, $\tau(s) = \alpha s$, the change of variable $s(x) = \cos x$ leads to

$$
\begin{aligned}
\tilde{V}_{m+1}(x) &= \left({\alpha'_m}^2 + \alpha'_m\right)\operatorname{cosec}^2 x + \delta \operatorname{cotan} x - {\alpha'_m}^2 + m(m-\alpha-1) \\
\tilde{W}_{m,\gamma}(x) &= \alpha'_m \operatorname{cotan} x + \sin x\,\tfrac{(1+\cos x)^{-\alpha/2+m-1}(1-\cos x)^{-\alpha/2+m-1}}{\gamma + \int^{\cos x}(1+t)^{-\alpha/2+m-1}(1-t)^{-\alpha/2+m-1}dt} + \tfrac{\delta}{2m-\alpha-1} \\
\tilde{\lambda}_m &= m(m-\alpha-1) - \tfrac{\delta^2}{(2m-\alpha-1)^2}
\end{aligned}
$$

9. *Eckart type potential*

In the case $\sigma(s) = s^2 - 1$, $\tau(s) = \alpha s$, the change of variable $s(x) = \cosh x$ leads to

$$
\begin{aligned}
\tilde{V}_{m+1}(x) &= (\alpha_m^2 - \alpha_m)\operatorname{cosech}^2 x - \delta \operatorname{cotanh} x + \alpha_m^2 - m(m+\alpha-1) \\
\tilde{W}_{m,\gamma}(x) &= \alpha_m \operatorname{cotanh} x + \sinh x\,\tfrac{(\cosh x+1)^{\alpha/2+m-1}(\cosh x-1)^{\alpha/2+m-1}}{\gamma + \int^{\cosh \mathrm{x}}(t+1)^{\alpha/2+m-1}(t-1)^{\alpha/2+m-1}dt} + \tfrac{\delta}{2m+\alpha-1} \\
\tilde{\lambda}_m &= -m(m+\alpha-1) - \tfrac{\delta^2}{(2m+\alpha-1)^2}.
\end{aligned}
$$

10. *Hyperbolic Rosen-Morse type potential*

In the case $\sigma(s)=s^2+1$, $\tau(s)=\alpha s$, the change of variable $s(x)=\sinh x$ leads to

$$\begin{array}{l} \tilde{V}_{m+1}(x) = (-\alpha_m^2+\alpha_m)\,\mathrm{sech}^2 x - \delta\tanh x + \alpha_m^2 - m(m+\alpha-1) \\ \tilde{W}_{m,\gamma}(x) = \alpha_m \tanh x + \cosh x \, \frac{(\cosh x)^{2m+\alpha-2}}{\gamma+\int^{\sinh x}(t^2+1)^{\alpha/2+m-1}dt} + \frac{\delta}{2m+\alpha-1} \\ \tilde{\lambda}_m = -m(m+\alpha-1) - \frac{\delta^2}{(2m+\alpha-1)^2}. \end{array}$$

VI. CONCLUDING REMARKS

Most of the exactly solvable Schrödinger equations are directly related to some hypergeometric type operators, and most of the formulae occurring in the study of these quantum systems follow from a small number of mathematical results concerning the orthogonal polynomials and the corresponding associated special functions. Particularly, most of the factorizations used in quantum mechanics follow from factorizations concerning the hypergeometric type operators. It is more advantageous to analyze the hypergeometric type operators then the operators occurring in various applications to quantum mechanics. The study of hypergeometric type operators avoids the occurrence of duplicate results, offers us the possibility to analyze most of the known exactly solvable quantum systems together, in a unified way, and to extend certain results known in particular cases. For all the quantum systems exactly solvable in terms of associated special functions and for almost all their supersymmetric partners a corresponding superpotential is given by (55).

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